

The Role of Type III Factors in Quantum Field Theory*

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Abstract

One of von Neumann's motivations for developing the theory of operator algebras and his and Murray's 1936 classification of factors was the question of possible decompositions of quantum systems into independent parts. For quantum systems with a finite number of degrees of freedom the simplest possibility, i.e., factors of type I in the terminology of Murray and von Neumann, are perfectly adequate. In relativistic quantum field theory (RQFT), on the other hand, factors of type III occur naturally. The same holds true in quantum statistical mechanics of infinite systems. In this brief review some physical consequences of the type III property of the von Neumann algebras corresponding to localized observables in RQFT and their difference from the type I case will be discussed. The cumulative effort of many people over more than 30 years has established a remarkable uniqueness result: The local algebras in RQFT are generically isomorphic to the unique, hyperfinite type III_1 factor in Connes' classification of 1973. Specific theories are characterized by the net structure of the collection of these isomorphic algebras for different space-time regions, i.e., the way they are embedded into each other.

John von Neumann was the father of the Hilbert space formulation of quantum mechanics [1] that has been the basis of almost all mathematically rigorous investigations of the theory to this day. We start by recalling the main concepts and explaining some notations.

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The *observables* of a quantum system are self adjoint operators $A = A^*$ on a complex, separable Hilbert space \mathcal{H} . For mathematical convenience we consider only observables in the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on \mathcal{H} . If ψ is a normalized vector in \mathcal{H} , i.e., $\langle \psi, \psi \rangle = 1$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on \mathcal{H} , then the *expectation value* of an observable A is given by

$$\omega_\psi(A) = \langle \psi, A\psi \rangle. \quad (1)$$

This is a positive, linear functional of A with $\omega_\psi(\mathbf{1}) = 1$ and ω_ψ is referred to as the *state* defined by ψ . More general states are given by *density matrices* ρ :

$$\omega_\rho(A) = \text{tr}(\rho A) = \sum_i \lambda_i \langle \psi_i, A\psi_i \rangle \quad (2)$$

where $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ are the eigenvalues and ψ_i the normalized eigenvectors of the positive trace class operator ρ .

If $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ then its *commutant* is defined as

$$\mathcal{S}' = \{B \in \mathcal{B}(\mathcal{H}) : [A, B] = 0 \text{ for all } A \in \mathcal{S}\}. \quad (3)$$

This is always a subalgebra of $\mathcal{B}(\mathcal{H})$ and if \mathcal{S} is $*$ -invariant then so is \mathcal{S}' . Moreover, it is closed in the topology defined by the states. A *von Neumann algebra* \mathcal{M} is a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ that is equal to its double commutant, i.e.,

$$\mathcal{M} = \mathcal{M}''. \quad (4)$$

A basic lemma of von Neumann says that this is equivalent to the algebra being closed in the topology defined by the states.

In a series of four papers [2, 3, 4, 5] Murray and von Neumann studied special von Neumann algebras, called *factors*. These are the algebras \mathcal{M} such that

$$\mathcal{M} \vee \mathcal{M}' \equiv \{AB : A \in \mathcal{M}, B \in \mathcal{M}'\}'' = \mathcal{B}(\mathcal{H}), \quad (5)$$

i.e., $\mathcal{B}(\mathcal{H})$ is “factorized” into \mathcal{M} and its commutant, \mathcal{M}' . This is equivalent to

$$\mathcal{M} \cap \mathcal{M}' = \mathbb{C}\mathbf{1}, \quad (6)$$

i.e., the center of \mathcal{M} contains only multiples of the identity operator. The mathematical problem addressed by Murray and von Neumann was to classify all possibilities for such algebras.

This problem is motivated by questions of mathematical nature but also “several aspects of the quantum mechanical formalism strongly suggest the elucidation of this subject” [2]. One of these aspects is the division of a quantum system into two independent subsystems. In the simplest case this is achieved as follows. The Hilbert space is written as a tensor product,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \quad (7)$$

The observables of one system correspond to the (self adjoint) elements of $\mathcal{M} = \mathcal{B}(\mathcal{H}_1) \otimes \mathbf{1}$ and the other to the commutant $\mathcal{M}' = \mathbf{1} \otimes \mathcal{B}(\mathcal{H}_2)$. The observables of the total system are thus factorized:

$$\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2). \quad (8)$$

Such a factorization of the observable algebra of the total system based on a tensor product factorization of the underlying Hilbert space is, in the terminology of Murray and von Neumann, the **Type I** case. It is characterized by the existence of *minimal projections* in \mathcal{M} : If $\psi \in \mathcal{H}_1$ and $E_\psi = |\psi\rangle\langle\psi|$ is the corresponding projection on the one-dimensional subspace of \mathcal{H}_1 generated by ψ , then

$$E = E_\psi \otimes \mathbf{1} \in \mathcal{M} \quad (9)$$

is a *minimal projection*, i.e., it has no proper subprojections in \mathcal{M} .

The other extreme is the **Type III** case. Here for every non-zero projection $E \in \mathcal{M}$ there exists a $W \in \mathcal{M}$ which maps $E\mathcal{H}$ isometrically onto \mathcal{H} , i.e.

$$W^*W = \mathbf{1}, \quad WW^* = E. \quad (10)$$

Von Neumann himself knew only sporadic examples of type III factors and he may, in fact, not have been fully aware of their significance. He was apparently more attracted by the **Type II** case that lies between the two extremes just described: A type II factor has no minimal projections, but every non-zero projection E has a subprojection $F < E$ that is finite in the sense that $F' < F$, $W^*W = F$, $WW^* = F'$ implies $F' = F$. This case, although very interesting from a mathematical point of view, is less important in quantum field theory than the other two and will not be discussed further here.

A finer classification of type III factors, based on Tomita-Takesaki modular theory [6], was pioneered by A. Connes [7]. There is a continuum of nonequivalent types, denoted III_λ with $0 \leq \lambda \leq 1$, and the case III_1 turns out to be of particular importance for relativistic quantum physics.¹ The characteristic feature of type III_1 is that the spectrum of the Tomita-Takesaki modular groups is the whole of \mathbb{R} . Among the factors that are generated by an increasing family of finite dimensional subalgebras (such factors are called *hyperfinite*) there is, up to equivalence, only *one* factor of type III_1 . This uniqueness result is due to U. Haagerup [10].

The only case a student of quantum mechanics is likely to encounter in standard texts is the type I case and this is, in fact, perfectly adequate as long as one deals only with systems with a finite number of degrees of freedom.² Hence it is a natural to ask whether the other cases are not just mathematical curiosities that a physicist need not bother about.

It will be argued below that it *is* important to keep the other possibilities in mind, otherwise one can be led to false physical conclusions. Investigations on the foundations

¹Examples of factors of type III_λ with $0 < \lambda < 1$ were given already in 1967 by Powers [8] and of type III_1 by Araki and Woods in 1968 [9].

²This can be attributed to von Neumann's theorem on the uniqueness of representations of the canonical commutation relations for systems with a finitely many degrees of freedom.

of *relativistic quantum field theory* (RQFT) revealed already more than 40 years ago that the algebras generated by observables that are localized in bounded regions of space-time cannot be type I and that they are, in fact, generically of type III. This is far from being obvious but turns out to be a consequence of general physical requirements of RQFT. Another case where type I is excluded is the statistical physics of infinite systems with nonzero density [11]. In the present review we shall not discuss the latter but focus on the situation in RQFT.

The physical difference between the type I situation and the actual state of affairs in RQFT can be illustrated by an instructive *gedankenexperiment* due to E. Fermi [12]. The analysis of this experiment drew considerable attention about 10 years ago (see [13] and [14] which contain also references to earlier literature) and papers on this subject have continued to appear since then (e.g., [15, 16, 17]). The standpoint taken in [14] and which will be explained below is that the “causality problems” associated with the gedankenexperiment are due to the implicit assumption that the experiment can be described within a type I framework which is not legitimate in RQFT.

The experiment envisaged in [12] is as follows. One considers two atoms, a and b , that are separated by a distance R . At time $t = 0$ atom a is in its ground state while atom b is in an excited state. Due to the decay of atom b and the emitted radiation absorbed by atom a , the latter will at some time $t > 0$ be in an excited state with nonzero probability. If the effect of the decay of atom b does not propagate faster than light, the time t must be at least R/c , i.e., for $t < R/c$ the state of atom a should be unchanged.

Let us consider first an analysis of this experiment within a “type I framework” which at first sight might look reasonable. In this framework the observable algebra for the atoms plus the radiation field is the tensor product of three type I factors: For the atom a the algebra is $\mathcal{M}_a = \mathcal{B}(\mathcal{H}_a)$ on a Hilbert space \mathcal{H}_a , for atom b it is $\mathcal{M}_b = \mathcal{B}(\mathcal{H}_b)$ on a Hilbert space \mathcal{H}_b and for the radiation field it is $\mathcal{M}_c = \mathcal{B}(\mathcal{H}_c)$ on a Hilbert space \mathcal{H}_c . The observables for the whole system are then

$$\mathcal{B}(\mathcal{H}) = \mathcal{M}_a \otimes \mathcal{M}_b \otimes \mathcal{M}_c \quad (11)$$

on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$.

The initial state of system is

$$\omega_0 = \omega_a \otimes \omega_b \otimes \omega_c. \quad (12)$$

Here ω_a is the ground state of atom a , ω_b an excited state of atom b and ω_c the vacuum state of the radiation field, i.e. a state without photons. The state at time $t > 0$ is now

$$\omega_t(\cdot) = \omega_0(e^{itH} \cdot e^{-itH}) \quad (13)$$

where H is the Hamiltonian of the total system. In Fermi’s original model the information about the spatial situation, i.e., the separation R of the atom is encoded in the interaction part of the Hamiltonian, but this is not important for the present discussion.

Within this setting the probability to find atom a excited at time t is

$$P(t) = \omega_t(E) \quad (14)$$

with $E = (\mathbf{1}_a - |\psi_a\rangle\langle\psi_a|) \otimes \mathbf{1}_b \otimes \mathbf{1}_c$ where ψ_a is the ground state of a . If the signal from the decay of atom b propagates at most with the speed of light one might thus expect that $P(t) = 0$ for $t < R/c$. This, however, is at variance with an analyticity property that follows from a stability assumption about the Hamiltonian H . Namely, if there is a lower bound for the energy (i.e., if the Hamiltonian has semibounded spectrum) then the vector valued function $t \mapsto Ee^{itH}\phi$ can, for all $\phi \in \mathcal{H}$, be continued to an analytic function of t in the upper half plane, $\text{Im } t > 0$. From this follows [13]:

If $\omega_t(E) = 0$ for all t in some time interval, then it is identically zero.

Hence, if atom a becomes excited at all, then this happens *immediately* after $t = 0$. Does this mean that quantum mechanics predicts faster than light propagation? The answer to this question is definitely 'no'. The lesson is rather: The naive type I ideas behind the mathematical setup for the gedankenexperiment just described are flawed [14]. Before discussing this further we have to introduce some formalism.

A proper framework for analyzing causal links in space and time is relativistic quantum field theory. Its general principles were formulated in the setting of von Neumann algebras in the 60's by R. Haag, D. Kastler, H. Araki, H.J. Borchers and others, see [18] and [19]. The basic object of any concrete model in RQFT is a *net* of von Neumann algebras, $\mathcal{O} \mapsto \mathcal{M}(\mathcal{O})$, on a Hilbert space \mathcal{H} , labeled by subsets \mathcal{O} of (Minkowski) space-time \mathbb{R}^4 . It satisfies

- Isotony: $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subset \mathcal{M}(\mathcal{O}_2)$.
- Local commutativity: If \mathcal{O}_1 is space-like separated from \mathcal{O}_2 then $\mathcal{M}(\mathcal{O}_1) \subset \mathcal{M}(\mathcal{O}_2)'$.
- Additivity: $(\bigcup_{x \in \mathbb{R}^4} \mathcal{M}(\mathcal{O}_0 + x))'' = \mathcal{B}(\mathcal{H})$ for all open \mathcal{O}_0 .

Furthermore, there is a unitary representation $U(x, \Lambda)$ of the Poincaré group on \mathcal{H} (here $x \in \mathbb{R}^4$ is a translation of space-time and Λ a Lorentz transformation) such that

$$U(x, \Lambda)\mathcal{M}(\mathcal{O})U(x, \Lambda)^{-1} = \mathcal{M}(\Lambda\mathcal{O} + x). \quad (15)$$

The joint spectrum of the generators of $U(x, 1)$ (i.e., the *energy-momentum spectrum*) is a subset of the forward light cone, and there is a normalized vector $\Omega \in \mathcal{H}$ (*vacuum*), unique up to a phase factor, that satisfies

$$U(x, \Lambda)\Omega = \Omega \quad (16)$$

for all Poincaré transformations (x, Λ) .

These general properties define a structure that is at the same time surprisingly rich and tight. Further natural properties, that can be verified in special models, restrict it further and lead in particular to a remarkable result, obtained through the work of several people in the course of more than 30 years :

The local algebras $\mathcal{M}(\mathcal{O})$ (with \mathcal{O} open and bounded) are, under physically plausible assumptions, the same for all RQFTs, namely they are isomorphic to the unique hyperfinite type III₁ factor.

Let us note some consequences of the type III property. First, since every projection $E \in \mathcal{M}(\mathcal{O})$ can be written as WW^* with an isometry $W \in \mathcal{M}(\mathcal{O})$ the mathematical structure is consistent with the idea that we can change *any* state ω into an eigenstate of E by a specific local operation without disturbing it in the causal complement:

Define $\omega_W(A) = \omega(W^*AW)$. Then

$$\omega_W(E) = \omega(W^*WW^*W) = \omega(\mathbf{1}) = 1 \quad (17)$$

but

$$\omega_W(B) = \omega(W^*BW) = \omega(W^*WB) = \omega(B) \quad (18)$$

for $B \in \mathcal{M}(\mathcal{O})'$.

The more refined type III₁ property implies even more: Every state φ can be approximated by ω_W for a *fixed* ω and suitable W , arbitrarily well in the norm topology. In other words: *Every state can be prepared locally, with arbitrary precision, from any other state.* Closely related is the fact that a local observable algebra $\mathcal{M}(\mathcal{O})$ has no pure states, i.e., for every ω there are ω_1 and ω_2 , different from ω , such that

$$\omega(A) = \frac{1}{2}\omega_1(A) + \frac{1}{2}\omega_2(A) \quad (19)$$

for all $A \in \mathcal{M}(\mathcal{O})$. On the other hand *every state on $\mathcal{M}(\mathcal{O})$ is a vector state*, i.e., for every ω there is a (non-unique!) $\psi_\omega \in \mathcal{H}$ such that

$$\omega(A) = \langle \psi_\omega, A\psi_\omega \rangle \quad (20)$$

for all $A \in \mathcal{M}(\mathcal{O})$.

After this interlude let us now return to Fermi's gedankenexperiment in light of the differences between the type I and type III situations. The analysis of [13], that seemed to indicate superluminal propagation, was independent of any details of the model and rested entirely on two assumptions: Stability, i.e., a lower bound for the energy, and the formula (14) for the excitation probability of atom a . While the stability assumption is not put to question the interpretation of (14) as an effect of atom b turns out to be at variance with basic principles of RQFT. A correct measure of the effect that takes these principles into account is perfectly causal.

In order to discuss the experiment within the general framework of RQFT (but independently of any specific model) one replaces the observable algebra \mathcal{M}_a for atom a by a local algebra $\mathcal{M}(\mathcal{O}_a)$. Here $\mathcal{O}_a \subset \mathbb{R}^4$ is a of the form $\mathcal{R}_a \times \{0\}$ where $\mathcal{R}_a \subset \mathbb{R}^3$ is a small ball where atom a (more precisely, its center of mass) is localized.³ We also include in the algebra $\mathcal{M}(\mathcal{O}_a)$ the observables of the electromagnetic field in \mathcal{O}_a (i.e., a

³Strictly speaking one should replace $\{0\}$ by a small time interval that takes into account the possibility that the observables might need a smearing in time to be well defined. This would lead to the appearance of a few ε 's in the formulas below but otherwise this point is not significant and for simplicity of notation we shall ignore it.

part of what was previously denoted \mathcal{M}_c). In the same way we replace \mathcal{M}_b (and part of \mathcal{M}_c) by a local algebra $\mathcal{M}(\mathcal{O}_b)$ with $\mathcal{O}_b = \mathcal{R}_b \times \{0\}$ where the ball \mathcal{R}_b is a distance R apart from \mathcal{R}_a .

The first thing to realize is that in order to measure a possible effect in \mathcal{R}_a of the decay of atom b in \mathcal{R}_b one must *compare* two states of the total system. In the first state, denoted ω_0 , atom b is at $t = 0$ present in \mathcal{R}_b and in an excited state. The second state, denoted $\omega_0^{(0)}$, describes the situation where atom b is in its ground state (or absent). In both cases the situation outside \mathcal{R}_b is at $t = 0$ is given by the ground state of atom a (with center of mass in \mathcal{R}_a). In other words: The states ω_0 and $\omega_0^{(0)}$ can, at $t = 0$, not be distinguished from the ground state of a by measurements outside of \mathcal{R}_b but they differ with respect to measurements inside \mathcal{R}_b .⁴ With time the states evolve, respectively, into ω_t and $\omega_t^{(0)}$. (For the sake of comparison with [13] we use here the Schrödinger picture.)

What does it mean to say that the state ω_t looks like $\omega_t^{(0)}$ in \mathcal{R}_a , i.e., the situation in \mathcal{R}_a is still unexcited after a time span $t > 0$? The answer is straightforward: The state ω_t should be indistinguishable from $\omega_t^{(0)}$ for all measurements carried out in \mathcal{R}_a which means that

$$\omega_t(A) = \omega_t^{(0)}(A) \text{ for all } A \in \mathcal{M}(\mathcal{O}_a). \quad (21)$$

The proper measure for the effect of the decay of atom b on the state in \mathcal{R}_a is thus obtained by considering the *deviation* of ω_t from $\omega_t^{(0)}$ with respect to all measurements that can be made in \mathcal{R}_a , i.e.,

$$D(t) = \sup_{A \in \mathcal{M}(\mathcal{O}_a); \|A\| \leq 1} \left| \omega_t(A) - \omega_t^{(0)}(A) \right|. \quad (22)$$

At $t = 0$ we not only have $D(0) = 0$ but, by the definition of ω_0 and $\omega_0^{(0)}$, also $\omega_0(A) - \omega_0^{(0)}(A) = 0$ for all A localized at a distance $< R$ from \mathcal{R}_a . In relativistic quantum field theories satisfying the axiom of primitive causality [21]⁵ this implies that $D(t)$ vanishes for $t < R/c$ where R is the spatial distance between \mathcal{R}_a and \mathcal{R}_b . (See Figure 1.) Hence there is no causality violation.

In the type I scenario, on the other hand, the excitation probability was written as $P(t) = \omega_t(E)$ with a projection E and this expression cannot vanish on an interval unless it vanishes identically. But since the observable algebra $\mathcal{M}(\mathcal{O}_a)$ is type III *there simply is no projection* whose non-vanishing expectation value in ω_t implies a non-vanishing $D(t)$.

It is instructive to elaborate a little more on this point, in particular in view of the fact that ω_0 restricted to $\mathcal{M}(\mathcal{O}_a)$ is given by a (non-unique) vector $\psi \in \mathcal{H}$, cf. Eq.(20).

⁴It may be necessary to leave the state unspecified in a small neighborhood of the boundary of \mathcal{R}_b , cf. the subsequent discussion of the "split property". But since this neighbourhood can be arbitrarily small this point is not relevant for the causality issue.

⁵In the present context this means the following. Let $\mathcal{O} = \mathcal{R} \times \{0\}$ with $\mathcal{R} \subset \mathbb{R}^3$. Then the von Neumann algebra $\mathcal{M}(\mathcal{O})$ is the same as the algebra $\mathcal{M}(\hat{\mathcal{O}})$ where $\hat{\mathcal{O}}$ is the causal closure of \mathcal{O} . These are the space-time points that are space-like with respect to all points that are space-like with respect to \mathcal{O} .

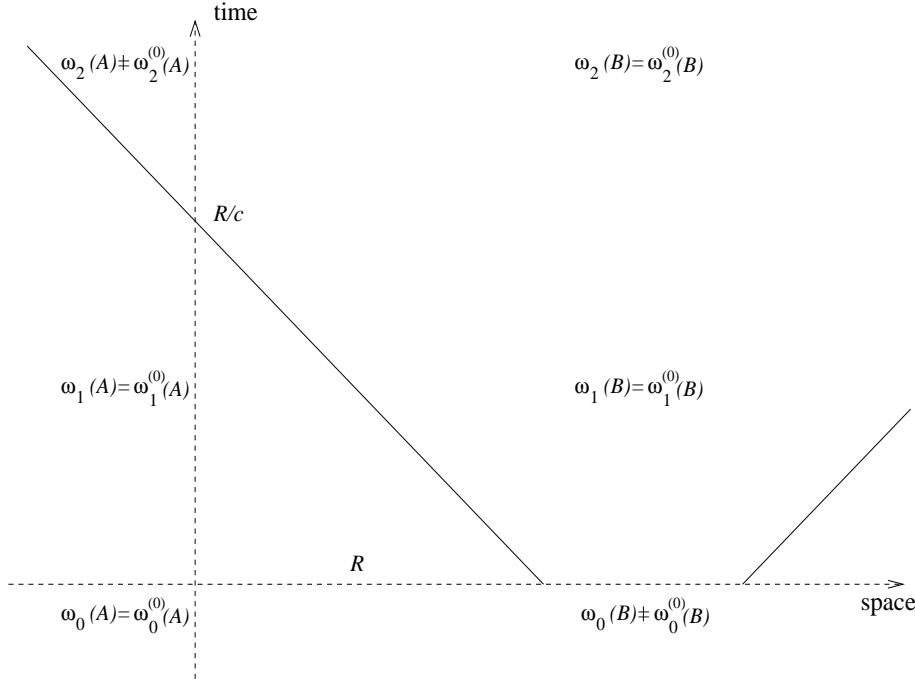


Figure 1: Space-time diagram illustrating that the situation in \mathcal{R}_b at $t = 0$ does not influence \mathcal{R}_a for $t < R/c$.

The projection on the orthogonal complement of ψ , $E = \mathbf{1} - |\psi\rangle\langle\psi|$, might appear to be a candidate for an operator to test the “excitation” in region \mathcal{R}_a . This, however, is in error because E is not in $\mathcal{M}(\mathcal{O}_a)$ and hence $D(t)$ can well be zero even if $\omega_t(E) > 0$. The type I situation is different because there one can take $E \in \mathcal{M}_a$ and $\omega_t^{(0)}(E) = 0$ for all t . But if nothing is known about the localization of E , even a nonvanishing difference $\omega_t(E) - \omega_t^{(0)}(E)$ does not imply any causality violation.

As will be discussed further below, it can in fact be expected that E belongs to some $\mathcal{M}(\mathcal{O})$ with \mathcal{O} slightly larger than \mathcal{O}_a . Then causality requires that $\omega_t(E) - \omega_t^{(0)}(E) = 0$ for $t < R'/c$ for some R' slightly smaller than R . This is perfectly compatible with the principles of RQFT. It is important to note that the purported “cause” for $\omega_t(E) > 0$, i.e., the initial excitation of atom b , is not present in the state $\omega_0^{(0)}$, by definition of that state. But if $\omega_t(E) > 0$ and $\omega_t(E) - \omega_t^{(0)}(E) = 0$ then also $\omega_t^{(0)}(E) > 0$. Hence a nonvanishing expectation value $\omega_t(E)$ can certainly not be attributed to a decay of atom b .

The last point becomes even more obvious if we simplify the experiment by dropping a altogether and take $\omega_0^{(0)}$ to be the vacuum state which is time invariant. Since $\omega_t(E) - \omega_0^{(0)}(E) = 0$ for $t < R'/c$ we see that $\omega_t(E)$ is equal to the constant $\omega_0^{(0)}(E) > 0$ for $t < R'/c$.⁶ The fact that the vacuum expectation value of any local, nonzero

⁶Note that, in contrast to the vector valued function $Ee^{-itH}\phi$, the expectation value $\omega_t(E)$ can *not* be analytically continued to complex t (it involves both e^{-itH} and e^{itH}) and can therefore well be constant on a whole interval without being constant everywhere.

projection is > 0 is a consequence of the Reeh-Schlieder theorem [20] which states that the vacuum vector (or more generally, any analytic vector for the energy) is cyclic and separating for all $\mathcal{M}(\mathcal{O})$. It is clearly not reasonable to attribute such vacuum fluctuations to some acausal propagation of signals.

We now leave the discussion of Fermi's gedankenexperiment and turn to the general structure of the local algebras of RQFT, starting with a brief account of some historic milestones.

After early discussions on the structure of local algebras by Haag and Schroer (1962) [21], Kadison (1963) [22] and Guenin and Misra (1963) [23], Araki proved in (1963–1964) [24, 25, 26] that the local algebras for relativistic free fields are type III factors. This establishes the same property also for interacting fields with local algebras equivalent to those of the free field (i.e., theories satisfying *local normality*). The relativistic structure is crucial here because in non-relativistic quantum field theory the local algebras are type I.

An important general result that *almost* establishes the type III property using only the basic premises of RQFT was obtained by Borchers (1967) [27]: For $\mathcal{O} \subset \mathbb{R}^4$ open and bounded and $\varepsilon > 0$ define

$$\mathcal{O}_\varepsilon = \bigcup_{|x| < \varepsilon} (\mathcal{O} + x). \quad (23)$$

Then every projection $E \in \mathcal{M}(\mathcal{O})$ can be written as WW^* with $W \in \mathcal{M}(\mathcal{O}_\varepsilon)$ and $W^*W = \mathbf{1}$.

In 1967 Powers gave explicit examples of a continuum of nonequivalent type III factors, derived from quantum statistical mechanics. These were the first new examples of type III factor since von Neumann's paper [4]. Araki and Woods took in 1968 the first steps towards the finer classification of type III factors and gave an example of type III_1 . This development culminated in Connes's classification in 1973 [7] based on Tomita-Takesaki modular theory.

For the algebras corresponding to space-like wedges (these are unbounded domains, generated by two light rays) Driessler [28, 29] used modular theory to prove that they are of type III_1 . See also [30] and [31] for more general results along these lines. For the wedge algebras the fundamental papers of Bisognano and Wichmann (1975–1976) [32, 33] are very relevant because they allow to identify the modular group defined by the vacuum state with the Lorentz boosts. Building on these results but assuming in addition asymptotic scale invariance, Fredenhagen proved in 1985 [34] that also algebras corresponding to bounded domains (double cones) are type III_1 .

In 1987 Haagerup established the uniqueness of the hyperfinite type III_1 factor. Buchholz, D'Antoni, Fredenhagen showed in 1987 [36] that natural assumptions on the size of sets of states that are approximately localized in phase space (*nuclearity properties*) lead to hyperfiniteness of the local algebras. Finally, Buchholz and Verch [35] in 1996 carried out a comprehensive analysis of the concept of scaling limits in RQFT and showed that if a model has a non-trivial scaling limit at all, then the local algebras are type III_1 .

The results of [36] on the hyperfiniteness are related to a more general analysis of the concept of *causal independence* (also called *statistical independence*) that sheds further light on the type III property of local algebras. The review article [38] contains a very thorough discussion of this issue and we shall here only touch upon a few points. Let us first define the relevant concepts.

Definitions. 1) A pair of commuting von Neumann algebras, \mathcal{M}_1 and \mathcal{M}_2 , in a common $\mathcal{B}(\mathcal{H})$ is *causally (statistically) independent* if for every pair of states, ω_1 on \mathcal{M}_1 and ω_2 on \mathcal{M}_2 , there is a state ω on $\mathcal{M}_1 \vee \mathcal{M}_2$ such that

$$\omega(AB) = \omega_1(A)\omega_2(B) \quad (24)$$

for $A \in \mathcal{M}_1$, $B \in \mathcal{M}_2$. In other words: States can be *independently* prescribed on \mathcal{M}_1 and \mathcal{M}_2 and extended to a common, *uncorrelated* state on the joint algebra. (This is really von Neumann's original concept of independent systems.)

2) We say that the pair \mathcal{M}_1 and \mathcal{M}_2 has the *split property* if there is a type I factor \mathcal{M} such that

$$\mathcal{M}_1 \subset \mathcal{M} \subset \mathcal{M}'_2. \quad (25)$$

Another way to state (25) is: There is a tensor product decomposition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1) \otimes \mathbf{1}, \quad \mathcal{M}_2 \subset \mathbf{1} \otimes \mathcal{B}(\mathcal{H}_2). \quad (26)$$

In the field theoretic context causal independence and split property for local algebras belonging to space-like separated domains are essentially equivalent. More precisely, the split property clearly implies statistical independence (one can take $\omega = \omega_1 \otimes \omega_2$), and conversely, statistical independence implies split property for algebras belonging possibly to slightly smaller domains than the original algebras [37].

Clearly it is desirable to have causal independence (and hence split property) if $\mathcal{M}_1 = \mathcal{M}(\mathcal{O})$ with \mathcal{O} some bounded domain and \mathcal{M}_2 is generated by the local algebras $\mathcal{M}(\tilde{\mathcal{O}})$ with $\tilde{\mathcal{O}}$ causally separated from \mathcal{O} by some positive distance. For instance, in Fermi's gedankenexperiment it is implicitly taken for granted that the state in \mathcal{R}_b at $t = 0$ can be specified independently of that in the complement of \mathcal{R}_b (with the exclusion of a small neighborhood of the boundary, cf. footnote 4). Hence type I factors play a very important role in RQFT. It is important to note, however, that the idealization of *strict* localization still requires the local algebras to be type III, even if they can be nested into commuting type I factors, cf. (25), with a slightly "fuzzy" localization. (Note that $\mathcal{B}(\mathcal{H}_1) \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{B}(\mathcal{H}_2)$ are not themselves local algebras although they can be included in local algebras belonging to slightly larger domains.)

We shall now give an example of a physical application of the split property. It was mentioned previously in connection with (17)–(18) that the type III property has consequences for the local preparation of states with prescribed properties. Combined with the split property a strong version of (17)–(18) emerges, and since it is easy to prove we include the proof here. It is a simplified version of results from [39], [40] and [38].

Proposition (Strong local preparability). *For every state ω and every bounded \mathcal{O} there is an isometry $W \in \mathcal{M}(\mathcal{O}_\varepsilon)$ (with \mathcal{O}_ε defined by (23)) such that for an arbitrary state φ*

$$\varphi(W^*AW) = \omega(A)$$

for all $A \in \mathcal{M}(\mathcal{O})$, but

$$\varphi(W^*BW) = \varphi(B)$$

for all $B \in \mathcal{M}(\mathcal{O}_\varepsilon)'$. In words: Every state can be prepared locally from any other state, with an isometry that depends only on the state to be prepared.

Proof: The split property implies that we can write $\mathcal{M}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_1) \otimes \mathbf{1}$, $\mathcal{M}(\mathcal{O}_\varepsilon)' \subset \mathbf{1} \otimes \mathcal{B}(\mathcal{H}_2)$. By the type III property of $\mathcal{M}(\mathcal{O})$ we have $\omega(A) = \langle \xi, A\xi \rangle$ for $A \in \mathcal{M}(\mathcal{O})$, with $\xi = \xi_1 \otimes \xi_2$. Then $E = E_{\xi_1} \otimes \mathbf{1} \in \mathcal{M}(\mathcal{O}_\varepsilon)$. By the type III property of $\mathcal{M}(\mathcal{O}_\varepsilon)$ there is a $W \in \mathcal{M}(\mathcal{O}_\varepsilon)$ with $W^*W = \mathbf{1}$, $WW^* = E$. QED

In the context of C^* algebras a general result on statistical independence was obtained by Roos in 1974 [41]. Shortly afterward the split property was proved to hold for free fields by Buchholz in 1974 [37]. The concept was developed systematically in [42] and in [43] it was brought in connection with nuclearity properties that express the idea that the *local* density of states (measured in a suitable sense) does not increase too fast with the energy. This line of thought was developed further in [44, 45] and other papers, e.g. [46] and [47].

The upshot is that the split property is both physically motivated and has been proved in models. It is the basis for the hyperfiniteness of the local algebras and hence, in connection with [35], of the universal structure of the local algebras.

Conclusions. The general postulates of quantum mechanics and (special) relativity, together with assumptions about the existence of a scaling limit and bounds on the local density of states, imply a *unique* structure of the local algebras: They are isomorphic to the unique, hyperfinite type III₁ factor. Moreover, every one of these factors is contained in a type I factor that in turn is contained a local algebra of a slightly larger domain. This general structure has important physical implications. The specific properties of individual models of RQFT (particle structure, scattering cross sections, bound states, superselection charges...) on the other hand, are encoded in the *net structure*

$$\mathcal{O} \mapsto \mathcal{M}(\mathcal{O}) \tag{27}$$

i.e. in the way these isomorphic type III factors for different \mathcal{O} are embedded into each other.

A challenge for the future, transcending von Neumann's original program, is to classify all such nets. The recent work of Kawahigashi and Longo [48], where this task is carried out for local conformal nets in two space-time dimensions and with central charge $c < 1$, is an indication that this may not be an entirely utopian vision.

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